A Note on Nonnegative Rank Factorizations

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ABSTRACT

A nonnegative weakly monotone matrix of rank r has a nonnegative rank factorization if and only if it possesses an $r\times r$ monomial submatrix.

1. INTRODUCTION

All matrices in this paper are nonnegative. The basic notation and definitions are from [1]. For example $P_r^{m\times n}$ denotes the set of nonnegative $m\times n$ matrices of rank r. By a nonnegative rank factorization of a matrix $A\in P_r^{m\times n}$ we shall mean a factorization

$$A=BC \tag{1}$$

where $B\in P_r^{m\times r}$ and $C\in P_r^{r\times n}$. In the literature over the last decade, the concept of a nonnegative rank factorization has been shown to have important applications in the study of nonnegative generalized inverses of nonnegative matrices. See, for instance, [2], [3], [5], and [7]. Not every nonnegative matrix has a nonnegative rank factorization [8]. No simple general procedure exists (at least to the authors' knowledge) that will determine when a given nonnegative matrix has a nonnegative rank factorization. It is clear that a nonnegative rank factorization such as (1) is equivalent to the existence of a cone $B\mathbb{R}^r_+$ which contains the columns of A and where $B\in P_r^{m\times r}$. If A is singular, then there exist permutation matrices R and S such that

$$RAS=\begin{bmatrix} M & MQ \\ PM & PMQ\end{bmatrix} \qquad \text{where} \quad M\in P_r^{r\times r}.$$

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Thomas [8] has shown that A has a nonnegative full rank factorization if and only if there exists $N \in P_r^{r \times r}$ such that $[M\, MQ]\mathbb{R}^n_+ \subset N\mathbb{R}^r_+ \subset \{x \geqslant 0 : Px \geqslant 0\}$. However, the determination of the existence of this cone $N\mathbb{R}^r_+$ can be difficult.

2. WEAKLY MONOTONE MATRICES

A matrix $A \in \mathbb{R}^{m \times n}$ is weakly monotone if and only if $Ax \geqslant 0$ implies that $x \in N(A) + \mathbb{R}^n_+$ [4]. An equivalent condition for $A \in P^{m \times n}$ to be weakly monotone is that $R(A) \cap \mathbb{R}^m_+ = A\mathbb{R}^n_+$. That is, weakly monotone for $A \in P^{m \times n}$ is equivalent to the existence of nonnegative solutions of $Ax = b$ whenever $Ax = b$ is consistent and $b \geqslant 0$. The next theorem shows that a nonnegative weakly monotone matrix A with a nonnegative rank factorization $A = BC$ imposes a particularly simple structure on both B and C.

THEOREM 1. *If $A \in P_r^{m \times n}$ is weakly monotone and has a nonnegative rank factorization $A = BC$, then each column of B is proportional to a column of A and each row of C is proportional to a row of A.*

Proof. From $A = BC$ we immediately obtain $A\mathbb{R}^n_+ \subset B\mathbb{R}^r_+$. Since C has full row rank, it possesses a right inverse C^- such that $CC^- = I_r$. Again from $A = BC$, it follows that $AC^- = B$. This leads to the observation that $B\mathbb{R}^r_+ \subset R(A) \cap \mathbb{R}^m_+$. Now $B\mathbb{R}^r_+ \subset R(A) \cap \mathbb{R}^m_+$ and the weak monotonicity of A ensure that $B\mathbb{R}^r_+ \subset A\mathbb{R}^n_+$. Thus $A\mathbb{R}^n_+ = B\mathbb{R}^r_+$. The only possible extremal elements up to scalar multiples of the cone $A\mathbb{R}^n_+$ are $A^{(1)}, \dots, A^{(n)}$. But $B^{(1)}, \dots, B^{(r)}$ are linearly independent and so must also be extremal elements of $A\mathbb{R}^n_+$. Therefore each column of B is a scalar multiple of some column of A. Next we demonstrate that each row of C is proportional to a row of A. This is accomplished by showing that B contains an $r \times r$ monomial submatrix. Suppose that $By \geqslant 0$. Then $By = Ax$ for some x. So $x = u + p$, where $u \in N(A)$ and $p \in \mathbb{R}^n_+$, because A is weakly monotone. Next $By = Ap = BCp$ implies that $y = Cp \geqslant 0$, as $N(B) = \{0\}$. Therefore $By \geqslant 0$ implies that $y \geqslant 0$. Hence B is monotone [6], which in turn ensures that B has a nonnegative $\{1\}$-inverse. This enables us to assert that B contains an $r \times r$ monomial submatrix [3]. ■

It follows from the proof of Theorem 1 that if $A \in P_r^{m \times n}$ is weakly monotone and has a nonnegative rank factorization, then A contains an $r \times r$ monomial submatrix. Clearly A has a nonnegative rank factorization if it contains an $r \times r$ monomial submatrix. Therefore we have the following corollary.

COROLLARY 1. *A nonnegative weakly monotone matrix of* rank r *has a nonnegative rank factorization if and only if it has a* $r\times r$ *monomial submatrix.*

At this point we add that not all nonnegative weakly monotone matrices have a nonnegative rank factorization. The matrix

$$P=\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix},$$

which was shown in [8] not to have a nonnegative rank factorization, provides a ready example. It is not difficult to show that P is weakly monotone. Since it does not contain a 3×3 monomial submatrix, Corollary 1 ensures and independently verifies the known fact that it does not possess a nonnegative rank factorization.

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